Week 6 Worksheet 11B
Volumes, Solids of Rotation, Differential Equations

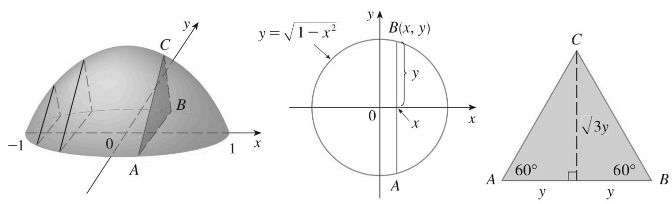
Definition of Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the **volume** of S is:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

(a) Cylinder $V = Ah$ (c) Box $V = lwh$

EXAMPLE 5 — Figure shows a solid with a circular base of radius 1. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume.



(a) The solid (b) Its base (c) A cross-section

SOLUTION: Since B lies on the circle, we have $y = \sqrt{1-x^2}$ and so the base of the triangle ABC is $|AB| = 2\sqrt{1-x^2}$. Since the triangle is equilateral, its height is

$$\sqrt{3}y = \sqrt{3}\sqrt{1-x^2}.$$

The cross-sectional area is therefore

$$A(x) = \frac{1}{2} \cdot 2\sqrt{1-x^2} \cdot \sqrt{3}\sqrt{1-x^2} = \sqrt{3}(1-x^2)$$

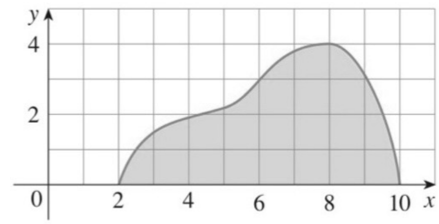
and the volume of the solid is

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \sqrt{3}(1-x^2)\,dx$$
$$= 2\int_0^1 \sqrt{3}(1-x^2)\,dx = 2\sqrt{3}\left[x - \frac{x^3}{3}\right]_0^1 = \frac{4\sqrt{3}}{3}$$

1. The height of a monument is 20 m. A horizontal cross-section at a distance x meters from the top is an equilateral triangle with side $\frac{1}{4}x$ meters. Find the volume.
2. The base of a solid is a circular disk with radius 3. Find the volume of the solid if parallel cross-sections perpendicular to the base are isosceles right triangles with hypotenuse lying along the base.
3. S is a right circular cone with height h and base radius r.
4. The base of S is a circular disk with radius r. Parallel cross-sections are squares.
5. The base of S is an elliptical region $9x^2 + 4y^2 = 36$. Cross-sections are isosceles right triangles with hypotenuse in base.

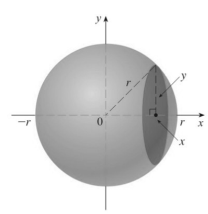
Solids of Revolution

6. If the region shown in the figure is rotated about the x-axis to form a solid, use the Midpoint Rule with $n = 4$ to estimate the volume.

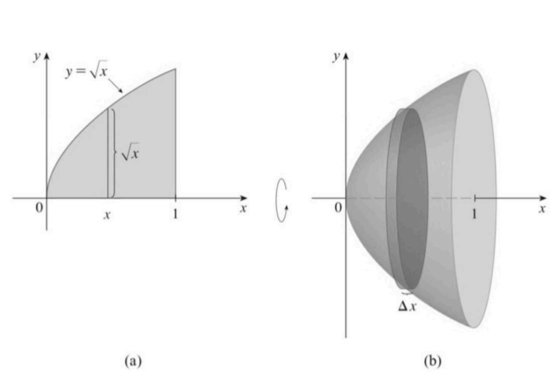


7. Estimate the volume if rotated about the y-axis using Midpoint Rule ($n = 4$).

8. Show that the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.



9. Volume for Solid?



10. Find the volumes of the solids obtained by rotating the region bounded by the curves $y = x$ and $y = x^2$ about the following lines. (a) The x-axis (b) The y-axis

11. Let $\mathcal{R}$ be the region bounded by the curves $y = 1 - x^2$ and $y = x^6 - x + 1$. Estimate the following quantities. (a) The x-coordinates of the points of intersection of the curves (b) The area of $\mathcal{R}$ (c) The volume generated when $\mathcal{R}$ is rotated about the x-axis

Differential Equations Introduction

Pure-time differential equations involve the derivative of the function but not the function itself. For example, if the rate of change of population size y depends on time only, this results in a differential equation of the form

$$\frac{dy}{dt} = f(t)$$

Autonomous differential equations arise when the equation involves both the derivative of the function and the function itself, but when there is no explicit dependence on the independent variable. Such equations have the general form

$$\frac{dy}{dt} = g(y)$$

where y is the unknown function of the independent variable t.

Nonautonomous differential equations are a combination of pure-time and autonomous differential equations. They arise when the equation involves the function and its derivative, and the independent variable appears explicitly as well.

1. Show that $y = \frac{2}{3}e^x + e^{-2x}$ is a solution of the differential equation $y' + 2y = 2e^x$. Is this differential equation pure-time, autonomous, or nonautonomous?

2. Verify that $y = -t\cos t - t$ is a solution of the initial-value problem
$$t\frac{dy}{dt} = y + t^2 \sin t \quad y(\pi) = 0$$
Is this differential equation pure-time, autonomous, or nonautonomous?

3. Show that $y = e^{-at}\cos t$ is a solution of the differential equation $y' = -e^{-at}(a\cos t + \sin t)$. Is this differential equation pure-time, autonomous, or nonautonomous?

4. (a) Show that every member of the family of functions $y = (\ln x + C)/x$ is a solution of the differential equation $x^2y' + xy = 1$.
(b) Illustrate part (a) by graphing several members of the family of solutions on a common screen.
(c) Find a solution of the differential equation that satisfies the initial condition $y(1) = 2$.
(d) Find a solution of the differential equation that satisfies the initial condition $y(2) = 1$.

6. (a) What can you say about the graph of a solution of the equation $y' = xy^3$ when x is close to 0? What if x is large?
(b) Verify that all members of the family $y = (C - x^2)^{-1/2}$ are solutions of the differential equation $y' = xy^3$.
(c) Graph several members of the family of solutions on a common screen. Do the graphs confirm what you predicted in part (a)?
(d) Find a solution of the initial-value problem
$$y' = xy^3 \quad y(0) = 2$$

13–15 Drug dissolution Differential equations have been used extensively in the study of drug dissolution for patients given oral medications. The three simplest equations used are the zero-order kinetic equation, the Noyes-Whitney equation, and the Weibull equation. All assume that the initial concentration is zero but make different assumptions about how the concentration increases over time during the dissolution of the medication.

13. **The zero-order kinetic equation** states that the rate of change in the concentration of drug c (in mg/mL) during dissolution is governed by the differential equation
$$\frac{dc}{dt} = k$$
where k is a positive constant. Is this differential equation pure-time, autonomous, or nonautonomous? State in words what this differential equation says about how drug dissolution occurs. What is the solution of this differential equation with the initial condition $c(0) = 0$?

14. **The Noyes-Whitney equation** for the dynamics of the drug concentration is
$$\frac{dc}{dt} = k(c_s - c)$$
where k and c_s are positive constants. Is this differential equation pure-time, autonomous, or nonautonomous? State in words what this differential equation says about how drug dissolution occurs. Verify that $c = c_s(1 - e^{-kt})$ is the solution to this equation for the initial condition $c(0) = 0$.

15. **The Weibull equation** for the dynamics of the drug concentration is
$$\frac{dc}{dt} = \frac{k}{t^b}(c_s - c)$$
where k, c_s, and b are positive constants and $b < 1$. Notice that this differential equation is undefined when $t = 0$. Is this differential equation pure-time, autonomous, or nonautonomous? State in words what this differential equation says about how drug dissolution occurs. Verify that
$$c = c_s(1 - e^{-\alpha t^{1-b}})$$
is a solution for $t \neq 0$, where $\alpha = k/(1-b)$.